

A separation in modulus property of the zeros of a partial theta function

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Abstract

We consider the partial theta function $\theta(q, z) := \sum_{j=0}^{\infty} q^{j(j+1)/2} z^j$, where $z \in \mathbb{C}$ is a variable and $q \in \mathbb{C}$, $0 < |q| < 1$, is a parameter. Set $\alpha_0 := \sqrt{3}/2\pi = 0.2756644477\dots$. We show that, for $n \geq 5$, for $|q| \leq 1 - 1/(\alpha_0 n)$ and for $k \geq n$ there exists a unique zero ξ_k of $\theta(q, \cdot)$ satisfying the inequalities $|q|^{-k+1/2} < |\xi_k| < |q|^{-k-1/2}$; all these zeros are simple ones. The moduli of the remaining $n - 1$ zeros are $\leq |q|^{-n+1/2}$. A *spectral value* of q is a value for which $\theta(q, \cdot)$ has a multiple zero. We prove the existence of the spectral values $0.4353184958\dots \pm i 0.1230440086\dots$ for which θ has double zeros $-5.963\dots \pm i 6.104\dots$

Keywords: partial theta function; separation in modulus; spectrum

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1 Introduction

The series $\theta(q, z) := \sum_{j=0}^{\infty} q^{j(j+1)/2} z^j$ in the variables q and z is convergent for $q \in \mathbb{D}_1 \setminus 0$, $z \in \mathbb{C}$ (here \mathbb{D}_a denotes the open disk centered at the origin and of radius a). The series defines a *partial theta function*. The terminology is explained by the fact that the Jacobi theta function is the sum of the series $\Theta(q, z) := \sum_{j=-\infty}^{\infty} q^{j^2} z^j$ and the equality $\theta(q^2, z/q) = \sum_{j=0}^{\infty} q^{j^2} z^j$ holds true; “partial” means that in the case of θ the sum is taken only on $\mathbb{N} \cup 0$, not on \mathbb{Z} . For any fixed value of the variable q (which we regard as a parameter), θ is an entire function in z .

The most recent application of the function θ is connected with a problem about hyperbolic polynomials (i.e. real polynomials having all their zeros real). It has been discussed in the articles [6], [7], [17] and [18]. These results are a continuation of an earlier study performed by Hardy, Petrovitch and Hutchinson (see [4], [5] and [19]). Other domains in which θ is used are statistical physics and combinatorics (see [20]), asymptotic analysis (see [2]), the theory of (mock) modular forms (see [3]) and Ramanujan-type q -series (see [22]). See more facts about θ in [1] and [20].

For $0 < |q| \leq 0.108$ all zeros of $\theta(q, \cdot)$ are distinct, see [9]. In fact, a stronger statement holds true. We say that, for fixed q , the zeros of θ are *separated in modulus* if one can enumerate these zeros in such a way that their moduli form a strictly increasing sequence tending to infinity (which implies that all zeros are simple). The following lemma is close to results formulated independently by A. Sokal and J. Forsgård; in [11] it has been formulated in a weaker version claiming only the absence of multiple zeros although the proof is the same:

Lemma 1. *For any $q \in \overline{\mathbb{D}_{c_0}}$, $c_0 := 0.2078750206\dots$, the zeros of the function θ are separated in modulus.*

Notation 2. For fixed q we denote by \mathcal{C}_k , $k \in \mathbb{N}$, the circumference in the z -space $|z| = |q|^{-k-1/2}$. To denote the restriction to \mathcal{C}_k of a given function in two variables we use the subscript k (e. g. θ_k stands for $\theta|_{\mathcal{C}_k}$).

Proof of Lemma 1. Consider for fixed q the function θ restricted to each of the circumferences \mathcal{C}_k , $k \in \mathbb{N}$. Fix k . Then in the series of θ the term of largest modulus is $L := z^k q^{k(k+1)/2}$ (one has $|L|_{|z|=|q|^{-k-1/2}} = |q|^{-k^2/2}$). The sum M of the moduli of all other terms is smaller than $|q|^{-k^2/2} \tau(|q|)$, where $\tau := 2 \sum_{\nu=1}^{\infty} |q|^{\nu^2/2}$. Indeed,

$$\begin{aligned} M + |L| &= \sum_{j=0}^{\infty} |q|^{j(j+1)/2 - j(k+1/2)} &= |q|^{-k^2/2} \sum_{j=0}^{\infty} |q|^{(j-k)^2/2} \\ &= |q|^{-k^2/2} (1 + 2 \sum_{\nu=1}^k |q|^{\nu^2/2} + \sum_{\nu=k+1}^{\infty} |q|^{\nu^2/2}) < |q|^{-k^2/2} (1 + \tau(|q|)) . \end{aligned} \quad (1)$$

The condition $1 \geq \tau(|q|)$ is tantamount to $|q| \leq c_0$. Thus for $|q| \leq c_0$ one has $|L| > M$.

One can also observe that the circumferences $|z| = |q|^{-k-1/2}$ separate the zeros of θ in the sense that no zero of θ lies on any of these circumferences for $|q| \leq c_0$. As we mentioned above, for $|q| \leq 0.108$ all zeros ξ_k of θ are simple. For any k fixed and for $|q|$ close to 0 one has $\xi_k \sim -q^{-k}$ (see Proposition 10 in [7]). Hence for $k \in \mathbb{N}$ and $|q| \leq c_0$ one has

$$|q|^{-k+1/2} < |\xi_k| < |q|^{-k-1/2} , \quad (2)$$

i.e. exactly one zero of θ lies between these two circumferences and all zeros are separated in modulus. One can continue analytically the zeros for $|q| \leq c_0$ and extend the inequalities (2) to the domain $\mathbb{D}_{c_0} \setminus 0$. Thus the enumeration of the zeros of θ given by the increasing of the modulus is valid in $\mathbb{D}_{c_0} \setminus 0$. \square

Remark 3. Using the same reasoning as the one in the proof of the lemma one can deduce that for $|q| \leq 0.2247945929 \dots$ the inequality $|\xi_1| < |q|^{-3/2}$ holds true. To this end one has to consider instead of the condition $1 \geq \tau(|q|)$ the inequality $1 \geq 2 \sum_{\nu=1}^k |q|^{\nu^2/2} + \sum_{\nu=k+1}^{\infty} |q|^{\nu^2/2}$ for $k = 1$, see (1).

Definition 4. In what follows we say that, for a given q , *strong separation* of the zeros of θ occurs for $k \geq k_0$ in the sense that for any $k \geq k_0$ there exists a unique zero ξ_k of θ which is simple and which satisfies condition (2).

For certain values of q with $|q| > c_0$ (we call them *spectral values*) the function $\theta(q, \cdot)$ has multiple zeros. It has been established in [10] that for any fixed value of the parameter q , the function θ has at most finitely-many multiple zeros. For $q \in (0, 1)$ there exists a sequence of values of q , tending to 1, for which $\theta(q, \cdot)$ has double real negative zeros tending to $-e^\pi$, see [8]. When $q \in (-1, 0)$, there exist two such sequences tending to -1 for which the corresponding double values of $\theta(q, \cdot)$ tend to $\pm e^{\pi/2}$, see [12]. The spectral number $\tilde{q}_1 := 0.3092493386 \dots$ (which is the smallest positive one) is connected with hyperbolic polynomials that remain such when their highest degree monomial is deleted, see [6] and [17]. There is numerical evidence that there are infinitely-many complex (not real) spectral numbers as well.

We denote by $\alpha_0 := \sqrt{3}/2\pi = 0.2756644477 \dots$ the positive solution to the equation $-(2\pi^2/3)\alpha + 1/(2\alpha) = 0$. In what follows we use the fact that for $n \geq 4$ one has $1 - 1/(\alpha_0 n) > 0$. The circumferences \mathcal{C}_k being defined using half-integer exponents we replace in the formulation of the theorem below the condition $n \geq 4$ by the weaker condition $n \geq 5$.

Theorem 5. (1) For $n \geq 5$ and for $|q| \leq 1 - 1/(\alpha_0 n)$ strong separation of the zeros ξ_k of θ occurs for $k \geq n$.

(2) For these zeros ξ_k one has $|\xi_k| \geq 336.2792102 \dots$.

(3) For any fixed q , all zeros of θ whose moduli are $\geq 4.685636519 \dots \times 10^5$, are strongly separated in modulus.

(4) For $0 < |q| \leq 1/2$ strong separation of the zeros of θ occurs for $k \geq 4$.

The paper is organized as follows. Section 2 contains some remarks about the spectrum of θ . Section 3 contains the proof of Theorem 5. Section 4 contains some notation used in Section 5. The latter contains the formulation of Proposition 8 claiming the existence of certain spectral values of q in the disk $\mathbb{D}_{1/2}$. Proposition 8 is proved in Section 6 while Lemmas 10 and 12 used in its proof are proved in Section 7.

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2 Some remarks about the zeros and the spectrum of θ

(1) Part (3) of the theorem is an improvement of the basic result of [11]. The latter states that, for any $0 < |q| < 1$ and for $|z| \geq 8^{11} = 8589934592$, all zeros of θ are simple.

If q is real, i. e. $q \in (-1, 0) \cup (0, 1)$, then all coefficients of $\theta(q, \cdot)$ are real and a priori θ can have only real zeros and/or pairs of complex conjugate ones. Part (3) of the theorem implies that the moduli of the latter are $\leq 4.685636519 \dots \times 10^5$.

(2) For any $q \in (0, 1)$, the function $\theta(q, \cdot)$ has infinitely-many real negative zeros (and no positive ones), and the double zeros if any are the rightmost negative ones. They are local minima for θ . In [8] it is proved that, for $q \in (0, 1)$, the real positive spectral values of θ have the following asymptotic presentation: $\tilde{q}_s = 1 - (\pi/2s) + o(1/s)$, where $0 < \tilde{q}_1 < \tilde{q}_2 < \dots < 1$ (a more precise presentation is obtained in [15]). For any $\gamma \in (0, 1)$ one can enumerate all but finitely-many of the zeros of $\theta(q, \cdot)$, $q \in (0, \gamma)$, so that:

1. For $0 < q < \gamma < \tilde{q}_1$, $\theta(q, \cdot)$ has all zeros real, negative, distinct and enumerated in the decreasing order. For any fixed index k , the corresponding zero ξ_k is continuous in q .
2. When $q = \tilde{q}_s$, the zeros $\xi_{2s-1} < 0$ and $\xi_{2s} < 0$ coalesce and then give birth to a complex conjugate pair, see part (2) of Theorem 1 of [7]. For any fixed index $k \geq 2s + 1$, the zero ξ_k is continuous in q for $q \in (0, \tilde{q}_{s+1})$.

Confluence of two real zeros of θ takes place also at negative spectral values; the asymptotics of the moduli of the spectral values for $q \in (-1, 0)$ is of the form $1 - (\pi/8s) + o(1/s)$, see [12]. For s odd (resp. for s even) θ has negative double zeros which are its local minima (resp. positive double zeros which are its local maxima). For any $q \in (-1, 0)$ the function $\theta(q, \cdot)$ has infinitely-many positive and infinitely-many negative zeros. The negative double zeros are the rightmost negative real zeros and the positive double zeros are the second from the left positive real zeros.

(3) It is shown in [13] that the zeros of θ are expanded in Laurent series in the parameter q of the form

$$\xi_k = -1/q^k + (-1)^k q^{k(k-1)/2} (1 + \Phi_k(q)) , \quad (3)$$

where Φ_k is a Taylor series with integer coefficients (in [13] the zeros are denoted by $-\xi_k$). Stabilization properties of the coefficients of the series Φ_k are proved in [13] and [14].

(4) Part (1) of Theorem 5 and parts (2) and (3) of the present remarks imply that the radius of convergence of the series (3) is $\leq 1 - (\pi/k) + o(1/k)$. Indeed, this radius equals the distance from 0 to the nearest singularity of the right-hand side of (3). This singularity is not further from 0 than $\tilde{q}_{[(k+1)/2]}$ (where $[]$ stands for the integer part of), see part (2) of the present remarks. Hence a priori the statement of part (1) of Theorem 5 could be improved only by looking for an inequality of the form $|q| \leq 1 - 1/(\alpha_1 n)$ with a constant $\alpha_1 \in (\alpha_0, 1/\pi]$, but not of the form $|q| \leq 1 - \tau(n)$ with $\tau = o(1/n)$.

(5) Consider for $k \geq 5$ the function $q^k \xi_k = -1 + (-1)^k q^{k(k+1)/2} (1 + \Phi_k(q))$. It is holomorphic for $|q| < 1 - 1/(\alpha_0 k)$ and continuous for $|q| \leq 1 - 1/(\alpha_0 k)$. Hence $|q^k \xi_k| \leq |q|^{-1/2}$, see (2). The maximum of this modulus is attained for $|q| = 1 - 1/(\alpha_0 k)$, therefore $|q^k \xi_k| \leq (1 - 1/(\alpha_0 k))^{-1/2}$. By the Cauchy inequalities, if one sets $q^k \xi_k = \sum_{j=0}^{\infty} h_{k,j} q^j$, then one obtains the estimation $|h_{k,j}| \leq (1 - 1/(\alpha_0 k))^{-j-1/2}$.

(6) It is shown in [10] that for any fixed $q \in \mathbb{D}_1$, and for k sufficiently large, the function θ has a zero close to $-q^{-k}$; these are all but finitely-many of the zeros of θ . This result is complementary to the ones of parts (1) and (4) of Theorem 5.

(7) The function θ has no zeros for $|z| \leq 1/2|q|$ (hence no zero for $|z| \leq 1/2$), see [7]. On the other hand, the radius of the disk in the z -space centered at 0 in which θ has no zeros for any $q \in \mathbb{D}_1$ is not larger than $0.5616599824\dots$. Indeed, consider the series $\theta^1 := \theta(\omega, z)$, $\omega := e^{3i\pi/4}$, for $|z| < 1$. The sequence $\{\omega^{j(j+1)/2}\}$ being 8-periodic the sum θ^1 equals

$$\left(\sum_{j=0}^7 \omega^{j(j+1)/2} z^j \right) / (1 - z^8) .$$

The zero of least modulus of its numerator is a simple one and equals

$$z_0 := 0.337553312314574\dots + i 0.448909453205253\dots , \quad \text{with } |z_0| = 0.5616599824\dots .$$

Hence for $\rho \in (0, 1)$ sufficiently close to 1 the function $\theta(\rho e^{3i\pi/4}, \cdot)$ has a zero close to z_0 . This follows from the uniform convergence as $\rho \rightarrow 1^-$ of $\theta(\rho e^{3i\pi/4}, z)$ to $\theta(e^{3i\pi/4}, z)$ on any compact subdomain of the unit disk in the z -space.

3 Proofs

Proof of Theorem 5. It is well-known that all zeros of the Jacobi theta function Θ are simple (see [23] and Chapter X of [21]), so this is also the case of the function $\Theta^*(q, z) = \Theta(\sqrt{q}, \sqrt{q}z) = \sum_{j=-\infty}^{\infty} q^{j(j+1)/2} z^j$. The following property is known as the *Jacobi triple product* (see [23]):

$$\Theta(q, z^2) = \prod_{m=1}^{\infty} (1 - q^{2m})(1 + z^2 q^{2m-1})(1 + z^{-2} q^{2m-1}) .$$

It implies the identity

$$\Theta^*(q, z) = \prod_{m=1}^{\infty} (1 - q^m)(1 + z q^m)(1 + q^{m-1}/z) . \quad (4)$$

Clearly the zeros of $\Theta^*(q, z)$ are all the numbers $\mu_s := -1/q^s$, $s \in \mathbb{Z}$. In what follows we set

$$Q := \prod_{m=1}^{\infty} (1 - q^m) \quad , \quad U := \prod_{m=1}^{\infty} (1 + zq^m)$$

$$R := \prod_{m=1}^{\infty} (1 + q^{m-1}/z) \quad , \quad G := \sum_{j=-\infty}^{-1} q^{j(j+1)/2} z^j = \Theta^* - \theta$$

Thus $\Theta^* = QUR$. Obviously, for $|z| \geq 2$ and $|q| < 1$ one has

$$|G| \leq \sum_{j=1}^{\infty} |q|^{j(j-1)/2} / 2^j \leq \sum_{j=1}^{\infty} 1/2^j = 1 . \quad (5)$$

The following lemma is part of Lemma 4 in [11]:

Lemma 6. *Suppose that $|q| \leq 1 - 1/b$, $b > 1$, and $|z| > 1$. Then $|Q| \geq e^{(\pi^2/6)(1-b)}$ and $|R| \geq (1 - 1/|z|)e^{(\pi^2/6)(1-b)}$.*

In particular, for $b = \alpha n$, $\alpha > 0$, one obtains

$$|Q| \geq e^{(\pi^2/6)(1-\alpha n)} \quad \text{and} \quad |R| \geq (1 - 1/|z|)e^{(\pi^2/6)(1-\alpha n)} . \quad (6)$$

If in addition $|z| \geq 2$, then $|R| \geq e^{(\pi^2/6)(1-\alpha n)}/2$.

Lemma 7. *Suppose that $|z| = |q|^{-n-1/2}$. Then $|U| \geq |q|^{-n^2/2} e^{-(\pi^2/3)(\alpha n(\alpha n-1))^{1/2}}$.*

Proof. It follows from the definition of U that

$$\begin{aligned} |U| &\geq \prod_{m=1}^n (|q|^{-n-1/2+m} - 1) \prod_{m=1}^{\infty} (1 - |q|^{m-1/2}) \\ &= |q|^{-n^2/2} \prod_{m=1}^n (1 - |q|^{m-1/2}) \prod_{m=1}^{\infty} (1 - |q|^{m-1/2}) \\ &\geq |q|^{-n^2/2} (\prod_{m=1}^{\infty} (1 - |q|^{m-1/2}))^2 . \end{aligned} \quad (7)$$

Set $P := \prod_{m=1}^{\infty} (1 - |q|^{m-1/2})$, hence $|U| \geq |q|^{-n^2/2} P^2$. Taking logarithms one obtains

$$\begin{aligned} \ln P &= -\sum_{\nu=1}^{\infty} |q|^{\nu-1/2} - (1/2) \sum_{\nu=1}^{\infty} |q|^{2\nu-1} - (1/3) \sum_{\nu=1}^{\infty} |q|^{3\nu-3/2} - \dots \\ &= -|q|^{1/2}/(1 - |q|) - |q|/2(1 - |q|^2) - |q|^{3/2}/3(1 - |q|^3) - \dots \\ &= (-|q|^{1/2}/(1 - |q|))T , \end{aligned}$$

$$\text{where } T = 1 + |q|^{1/2}/2(1 + |q|) + |q|/3(1 + |q| + |q|^2) + \dots .$$

It is clear that $|q|^{s/2}/(s+1)(1 + |q| + \dots + |q|^s) < 1/(s+1)^2$ (because $|q|^r + |q|^{s-r} \geq 2|q|^{s/2}$, $r = 0, \dots, [s/2]$). Recall that $\sum_{s=0}^{\infty} 1/(s+1)^2 = \pi^2/6 = 1.6449\dots$. Hence $T \in (0, \pi^2/6)$ and $P \geq e^{-(\pi^2/6)|q|^{1/2}/(1-|q|)}$. Fix $\alpha > 0$. For $|q| \leq 1 - 1/\alpha n$ this implies $P \geq e^{-(\pi^2/6)(\alpha n(\alpha n-1))^{1/2}}$ from which the lemma follows. \square

For $|q| \leq 1 - 1/\alpha n$ the minoration $|q|^{-n^2/2} e^{-(\pi^2/3)(\alpha n(\alpha n-1))^{1/2}}$ of $|U|$ is not less than $(1 - 1/\alpha n)^{-n^2/2} e^{-(\pi^2/3)(\alpha n(\alpha n-1))^{1/2}}$. The quantity $(1 - 1/\alpha n)^{-n}$ is decreasing as n is increasing; it tends to $e^{1/\alpha}$ as n tends to infinity. Therefore $(1 - 1/\alpha n)^{-n^2/2} \geq e^{n/(2\alpha)}$ and $|U| \geq e^{n/(2\alpha)} e^{-(\pi^2/3)(\alpha n(\alpha n-1))^{1/2}} \geq e^{n/2\alpha - (\pi^2/3)\alpha n}$. Thus one obtains the estimation (see formula (4), conditions (6), the line that follows them and conditions (7))

$$\begin{aligned}
|\Theta^*| &= |Q||U||R| \geq e^{(\pi^2/6)(1-\alpha n)} e^{n/2\alpha - (\pi^2/3)\alpha n} (e^{(\pi^2/6)(1-\alpha n)}/2) \\
&= e^{(\pi^2/3) + (-(2\pi^2/3)\alpha + 1/(2\alpha))n} / 2.
\end{aligned}$$

For $\alpha = \alpha_0$ one gets $|\Theta^*| \geq e^{\pi^2/3}/2$. Recall that $\theta = \Theta^* - G$. For the restrictions θ_k , Θ_k^* and G_k of these functions to the circumference \mathcal{C}_k one has $|\Theta_k^*| \geq e^{\pi^2/3}/2$ and $|G_k| \leq 1$, therefore for $t \in [0, 1]$ one has $|\Theta_k^* - tG_k| \geq (e^{\pi^2/3}/2) - 1 > 0$. This means that no zero of the function $\Theta_k^* - tG_k$ crosses the circumference \mathcal{C}_k as t increases from 0 to 1. This is true for any $k \geq n$. Thus to prove part (1) of the theorem there remains to lift the condition $|z| \geq 2$ which was used to obtain the estimation $|R| \geq e^{(\pi^2/6)(1-\alpha n)}/2$.

Suppose that $k \geq n \geq 5$ and $|q| \leq 1 - 1/(\alpha_0 n)$. Then

$$|\xi_k| \geq (1 - 1/(n\alpha_0))^{-n+1/2} \geq (1 - 1/(5\alpha_0))^{-9/2} = 336.2792102 \dots$$

(We use the fact that the function $(1 - 1/(x\alpha_0))^{-x+1/2}$ is decreasing for $x \geq 5$.) This proves part (2) of the theorem and also lifts the restriction $|z| \geq 2$. Now part (1) of the theorem is completely proved.

Suppose that $1 - 1/(\alpha_0(n-1)) < |q| \leq 1 - 1/(\alpha_0 n)$. The zero ξ_n of θ is the one of smallest modulus among its zeros strongly separated in modulus which are mentioned in parts (1) and (2) of the theorem. One has

$$|\xi_n| \leq |q|^{-n-1/2} < (1 - 1/(\alpha_0(n-1)))^{-n-1/2}.$$

The right-hand side is maximal for $n = 5$; the corresponding value is $4.685636519 \dots \times 10^5$. This proves part (3) of the theorem.

To prove part (4) of the theorem we perform the same reasoning as above yet we use more accurate inequalities. In particular, we consider $|z|$ to be not less than $|q|^{-n-1/2}$ which for $n \geq 3$ and $|q| \leq 1/2$ implies $|z| \geq d_0 := 11.31370850 \dots$. This allows to make the estimations

$$\begin{aligned}
|R| &\geq \prod_{m=1}^{\infty} (1 - 2^{1-m}/d_0) =: r_0 = 0.8333799934 \dots, \\
P &\geq \prod_{m=1}^{\infty} (1 - 2^{1/2-m}) =: p_0 = 0.1298980722 \dots \quad \text{and} \\
|Q| &\geq \prod_{m=1}^{\infty} (1 - 2^{-m}) =: q_0 = 0.2887880952 \dots
\end{aligned}$$

Recall that (see (7)) $|U| \geq |q|^{-n^2/2} \prod_{m=1}^n (1 - |q|^{m-1/2})P$. For $0 < |q| \leq 1/2$ and $n \geq 3$ this product is minimal for $|q| = 1/2$ and $n = 3$ when it equals

$$u_0 := 2^{9/2} \times 0.1558689591 \dots \times p_0 = 0.4581390612 \dots$$

Thus one obtains the estimation

$$|\Theta^*| \geq |R||Q||U| \geq r_0 q_0 u_0 = 0.1102604290 \dots$$

On the other hand one has

$$|G| \leq \sum_{j=1}^{\infty} d_0^{-j} (1/2)^{j(j-1)/2} = 0.09213257671 \dots < 0.1102604290 \dots$$

As in the proof of Theorem 5 one concludes that for $0 < |q| \leq 1/2$ strong separation of the zeros of θ occurs for $k \geq 4$.

□

4 Notation and preliminary remarks

In this section we fix some notation which is to be used in next section. We set $\rho := 0.4353184958$, $\tau := 0.1230440086$. Observe that these are rational numbers; for infinite decimal fractions we write $0.4353184958\dots$ and $0.1230440086\dots$. We set $\varepsilon := 2 \times 10^{-10}$ and we denote by $U \subset \mathbb{C}$ the rectangle (in the q -space)

$$U := \left\{ \begin{array}{l} \operatorname{Re} q \in [\rho - \varepsilon, \rho + \varepsilon] \\ \operatorname{Im} q \in [\tau - \varepsilon, \tau + \varepsilon] \end{array} \right\} = [0.4353184956, 0.4353184960] \times [0.1230440084, 0.1230440088]$$

We define the rectangle $V \subset \mathbb{C}$ (in the z -space) as the set

$$V := \{ \operatorname{Re} z \in [-5.961, -5.965], \operatorname{Im} z \in [6.102, 6.106] \}.$$

We denote its vertices as follows:

$$A = (-5.965, 6.102), \quad B = (-5.961, 6.102), \quad C = (-5.961, 6.106), \quad D = (-5.965, 6.106).$$

By abuse of language we denote by A also the complex number $-5.965 + i 6.102$ etc.

In what follows we consider several functions which are defined after the function θ . The subscripts z and q mean partial derivations, e. g. $\theta_z := \partial\theta/\partial z$, $\theta_{qz} := \partial^2\theta/\partial q\partial z$ etc. Thus

$$\theta_{qz} = 1 + 6q^2z + 18q^5z^2 + 40q^9z^3 + 75q^{14}z^4 + 126q^{20}z^5 + 196q^{27}z^6 + \dots \quad (8)$$

The subscript (k) means truncation, i. e. $\theta_{(k)} := \sum_{j=0}^k q^{j(j+1)/2} z^j$.

We set $\theta^* := (1/2q^3)\theta_{zz}$. It is clear that

$$\theta^* = 1 + 3q^3z + 6q^7z^2 + 10q^{12}z^3 + 15q^{18}z^4 + 21q^{25}z^5 + 28q^{33}z^6 + \dots \quad (9)$$

For $5 \leq j \leq n \leq \infty$ we set $r_j \in [0, 1]$, $r := (r_5, r_6, \dots, r_n)$ if $n < \infty$ or $r := (r_5, r_6, \dots)$ if not. We define the family of functions

$$\theta_r(q, z) := 1 + qz + q^3z^2 + q^6z^3 + q^{10}z^4 + \sum_{j=5}^n r_j q^{j(j+1)/2} z^j.$$

For a function $f(q, z)$ defined on $U \times V$ we denote by $DR[f]$ (resp. $DI[f]$) the maximal possible absolute value of the difference between the values of $\operatorname{Re} f$ (resp. $\operatorname{Im} f$) at two different points of $U \times V$. Obviously, for two functions $f(q, z)$ and $g(q, z)$ one has

$$DR[f + g] \leq DR[f] + DR[g] \quad \text{and} \quad DI[f + g] \leq DI[f] + DI[g]. \quad (10)$$

5 The spectral values closest to 0

In the present section we consider the restriction of θ to the disk $\mathbb{D}_{1/2}$. We prove in the next section the following

Proposition 8. For $0 < |q| \leq 1/2$ the function θ has at least three spectral values which equal

$$\tilde{q}_1 := 0.3092493386\dots \quad \text{and} \quad v_{\pm} := 0.4353184958\dots \pm i 0.1230440086\dots$$

The function $\theta(\tilde{q}_1, \cdot)$ has a double real negative zero $-7.5032559833\dots$, all its other zeros are real, negative, simple and < -8 . The function $\theta(v_{\pm}, \cdot)$ has a simple zero $-3.277\dots \mp i 1.483\dots$, a double zero $-5.963\dots \pm i 6.104\dots$, its other zeros are simple and of modulus larger than $|v_{\pm}|^{-7/2} = 16.06050040\dots$, see part (4) of Theorem 5.

In the present section we present a hint why the following conjecture should be true:

Conjecture 1. The spectral values \tilde{q}_1 and v_{\pm} are the only spectral values of θ for $|q| \leq 1/2$.

(These spectral numbers are mentioned in the lectures of A. Sokal.)

Hint of a proof. One can approximate θ by its truncations $\theta_{(s)} := \sum_{j=0}^s q^{j(j+1)/2} z^j$. We consider $\theta_{(s)}$ as a degree s polynomial in z . The values of q for which the latter has multiple zeros are the values for which one has $\text{Res}(\theta_{(s)}, \partial\theta_{(s)}/\partial z, z) = 0$.

For $|q| \leq 1/2$, the truncations with $s = 13, \dots, 24$ have multiple zeros for $q \approx \tilde{q}_1$, for $q \approx v_{\pm}$ and for no other value of q . Up to the 10th decimal, these values of q are the same for $s = 13, \dots, 24$. This fact makes Conjecture 1 plausible, but does not provide a rigorous proof of it.

It is proved in [16] that \tilde{q}_1 is the only spectral value of θ belonging to $\mathbb{D}_{0.31}$.

Part (4) of Theorem 5 implies that for $0.31 \leq |q| \leq 0.5$ the multiple zeros of θ (if any) have a modulus $\leq 0.31^{-3.5} = 60.28844350\dots$. The first term of the series $\theta - \theta_{(18)}$ equals $q^{190} z^{19}$. For $|q| = 0.5$ and $|z| = 0.31^{-3.5}$ its modulus equals $4.253517108\dots \times 10^{-24}$. Therefore one expects that the truncation $\theta_{(18)}$ provides sufficient accuracy in the computation of the three spectral values of θ closest to 0. \square

Remarks 9. (1) The approximations up to the 6th decimal of the first 25 real positive spectral values are equal to (see [17])

$$\begin{array}{ccccccccc} 0.309249, & 0.516959, & 0.630628, & 0.701265, & 0.749269, & & & & \\ 0.783984, & 0.810251, & 0.830816, & 0.847353, & 0.860942, & & & & \\ 0.872305, & 0.881949, & 0.890237, & 0.897435, & 0.903747, & & & & \\ 0.909325, & 0.914291, & 0.918741, & 0.922751, & 0.926384, & & & & \\ 0.929689, & 0.932711, & 0.935482, & 0.938035, & 0.940393. & & & & \end{array}$$

(2) The approximations up to the 6th decimal of the moduli of the first 8 negative spectral values are equal to (see [12])

$$0.727133, 0.783742, 0.841601, 0.861257, 0.887952, 0.897904, 0.913191, 0.919201.$$

(3) One has $|v_{\pm}| = 0.4523737623\dots$. As we said above, the spectral value \tilde{q}_1 is the closest to 0. Of the other real spectral values closest to the border of $\mathbb{D}_{1/2}$ (and also to 0) is $w := 0.5169593598\dots$. It seems that it is the next closest to 0 (after v_{\pm}) among all spectral values because the next after w closest to 0 of the zeros of $\text{Res}(\theta_{(18)}, \partial\theta_{(18)}/\partial z, z)$ equal $0.5373389195\dots \pm i 0.1803273369\dots$. Their modulus equals $0.5667901400\dots$.

6 Proof of Proposition 8

The statements of the proposition concerning \tilde{q}_1 and the double zero $-7.5\dots$ are proved in [17] and [16]. The lemmas from this section are proved in the next one.

Lemma 10. *For $q \in U$ and $|z| = |q|^{-2}$, and for any r as in Section 4 one has $\theta_r(q, z) \neq 0$.*

Remark 11. The family of functions θ_r contains, in particular, the functions $\theta_{(18)}$ and θ . Lemma 10 implies that the smallest of the moduli of the zeros of any of the functions θ_r is less than $|q|^{-2}$ when $q \in U$. Indeed, the smallest modulus zero of $\theta_{(18)}(\rho + i\tau, \cdot)$ equals $-3.27794407050033\dots - i 0.148307531004121\dots$, its modulus is less than 4 while for $q \in U$ one has $5 < |q|^{-2}$; this can be checked numerically. There remains to apply a continuity argument.

By part (4) of Theorem 5, for $q \in U$ the function θ has two simple zeros or one double zero whose moduli belong to the interval $[|q|^{-2}, |q|^{-7/2}]$.

Lemma 12. (1) *For $(q, z) \in U \times V$ one has $\operatorname{Re}(\theta^*) \in (0.03, 0.08)$ and $\operatorname{Im}(\theta^*) \in (0.15, 0.20)$.*

(2) *For $(q, z) \in U \times V$ one has $\operatorname{Re}(\theta_{qz}) \in (-0.70, 0.84)$ and $\operatorname{Im}(\theta_{qz}) \in (-2.33, -0.79)$.*

For $q = \rho + i\tau$, the function $\theta_z(q, \cdot)$ has a zero which equals $-5.963\dots + i 6.104\dots$. This is a simple zero of $\theta_z(q, \cdot)$, see part (1) of Lemma 12. Hence it can be considered as a function $\eta(q)$ (as long as $q \in U$ and the values of this function belong to V).

Consider the level sets $\{\theta = \text{const}\}$. The function θ satisfies the equality

$$2q\theta_q = z((z\theta)_{zz}) = z^2\theta_{zz} + 2z\theta_z. \quad (11)$$

As $\theta_z = 0$ along the graph of η and as $\theta_{zz} \neq 0$ in $U \times V$ (see part (1) of Lemma 12), one deduces from (11) that $\theta_q \neq 0$ along the graph of η . Hence the level sets $\{\theta = \text{const}\}$ are locally analytic at their intersection points with this graph and their tangent spaces at these points are parallel to the z -space.

Differentiating the equality $\theta_z(q, \eta(q)) = 0$ w.r.t. q one gets $\eta_q = -\theta_{qz}/\theta_{zz} = -\theta_{qz}/(2q^3)\theta^*$. Lemma 12 implies that

$$|\eta_q| \leq (0.84^2 + 2.33^2)^{1/2}/2(q^*)^3(0.03^2 + 0.15^2)^{1/2} \leq 87.44992430\dots < 87.45, \quad (12)$$

where $q^* := (0.4353184956^2 + 0.1230440084^2)^{1/2} = 0.4523737621\dots$ is the smallest possible modulus of a number from U .

$$\begin{aligned} \text{Set } q_a &:= 0.4353184958244864 + i 0.1230440085519491, \\ z_a &:= -5.963923719619588 + i 6.104775174235743. \end{aligned}$$

One can check numerically that:

$$\begin{aligned} \theta(q_a, z_a) &= -1.6\dots \times 10^{-15} + i 2.8\dots \times 10^{-16} =: \chi_0, \\ \theta_z(q_a, z_a) &= -8.0\dots \times 10^{-16} + i 0.0\dots \times 10^{-15} =: \lambda^*. \end{aligned} \quad (13)$$

Consider for $q = q_a$ the vector field $\dot{z} = -1/\theta_{zz}$ (we denote the time by λ , i. e. $\dot{z} = dz/d\lambda$). Its phase curve which for $\lambda = 0$ passes through $z = z_a$, for $z = \lambda^*$ passes through a point z^* with $\theta_z(q_a, z^*) = 0$. As $|\theta_{zz}| \geq (0.03^2 + 0.15^2)^{1/2}$ (see part (1) of Lemma 12), one has

$$|z^* - z_a| \leq |\lambda^*|/(0.03^2 + 0.15^2)^{1/2} = 5.2\dots \times 10^{-15} \quad (14)$$

The restriction W of the subset $\{\theta_z = 0\}$ to the cartesian product $U \times V$ is locally a smooth complex curve. The set W is the graph of a function, continuous on U and analytic inside U .

Indeed, consider inequalities (12). Denote by (q', z') and (q'', z'') any two points of $U \times V$. The distance between any two points of U is $\leq 2\varepsilon$, therefore $|z' - z''| \leq (2\varepsilon) \times 87.45 = 3.498 \times 10^{-8}$. Set $z' := z^*$. The last inequality, combined with inequality (14) and the definition of z_a and V , implies that $z'' \in V$. Hence for any $q \in U$, the value of η belongs to the set V . Analyticity and continuity of the function $\eta(q)$ follow from the fact that η is a simple zero of θ_z .

Denote by $I \subset \mathbb{C}$ the segment with extremities at 0 and λ^* . The maximal possible value of $|\theta_z|$ at a point of the phase curve (with $\lambda \in I$) is not larger than

$$\left(\max_{U \times V} |\theta_{zz}| \right) \times |\lambda^*| \leq (0.08^2 + 0.20^2)^{1/2} \lambda^* = 1.7 \dots \times 10^{-16} =: \mu_0 .$$

One has $\theta(q_a, z_a) = \chi_0$. Therefore

$$|\theta(q_a, z^*)| \leq |\chi_0| + |z^* - z_a| \mu_0 < 1.625 \dots \times 10^{-15} . \quad (15)$$

Define a vector field on W , with time q , by the formulas $q_q = 1$, $z_q = -\theta_{qz}/\theta_{zz} = -\theta_{qz}/(2q/z^2)\theta_q$ (see (11)). Introduce as new time the value of θ . Hence the vectorfield is defined by the formulas

$$dq/d\theta = 1/\theta_q \quad , \quad dz/d\theta = -\theta_{qz}/(2q/z^2)(\theta_q)^2 . \quad (16)$$

Denote by q^\dagger the value of q (if it exists) for which the phase curve of this vector field with initial condition $(q, z) = (q_a, z^*)$ passes through a point (q^\dagger, \tilde{z}) such that $\theta(q^\dagger, \tilde{z}) = 0$. We want to show that $q^\dagger \in U$ (hence $(q^\dagger, \tilde{z}) \in U \times V$) which implies that this value of q indeed exists.

For $(q, z) \in W$ one has $\theta_q = q^2 z^2 \theta^*$ (this follows from (11) and the definition of θ^*). The extremal values of $\arg(q^2 z^2)$ (for $(q, z) \in U \times V$) are obtained for

$$q = \rho + \varepsilon + i(\tau - \varepsilon), \quad z = C \quad \text{and} \quad q = \rho - \varepsilon + i(\tau + \varepsilon), \quad z = A .$$

They equal respectively

$$-1.043893693643218 \dots \quad \text{and} \quad -1.042567942295371 \dots .$$

The extremal values of $|q^2 z^2|$ are obtained for

$$q = \rho - \varepsilon + i(\tau - \varepsilon), \quad z = B \quad \text{and} \quad q = \rho + \varepsilon + i(\tau + \varepsilon), \quad z = D .$$

They are equal to

$$3.858934465358369 \dots \quad \text{and} \quad 3.861493307333390 \dots .$$

By part (1) of Lemma 12 one has

$$\begin{aligned} \arg(\theta^*) &\in [\quad 1.080839000541168 \dots \quad , \quad 1.421906379185399 \dots \quad] \\ |\theta^*| &\in [\quad 0.1529705854077835 \dots \quad , \quad 0.2154065922853802 \dots \quad] \end{aligned}$$

Thus

$$\begin{aligned} \arg(q^2 z^2 \theta^*) &\in [\quad 0.036945306897950 \dots \quad , \quad 0.379338436890028 \dots \quad] \\ |q^2 z^2 \theta^*| &\in [\quad 0.5903034642161417 \dots \quad , \quad 0.8317911144654879 \dots \quad] \end{aligned}$$

and

$$\begin{aligned} \arg(1/\theta_q) &\in [\quad -0.379338436890028 \dots \quad , \quad -0.036945306897950 \dots \quad] \\ |1/\theta_q| &\in [\quad 1.202224912732572 \dots \quad , \quad 1.694043929299806 \dots \quad] \end{aligned} \quad (17)$$

The quantity q^\dagger is obtained by integrating the value of $1/\theta_q$ when the value of θ runs over the segment with extremities $\theta(q_a, z^*)$ and 0. Inequalities (15) and (17) imply that $|q^\dagger - q_a| < 10^{-10}$ hence $q^\dagger \in U$.

7 Proofs of Lemmas 10 and 12

In the proofs of Lemmas 10 and 12 we use the following example:

Example 13. Consider the monomial $10q^{12}z^3$ (this is the fourth monomial of θ^* , see (9)). Set

$$\delta := |2(1+i)\varepsilon/(\rho+i\tau-(1+i)\varepsilon)| = 1.250482394\dots \times 10^{-9}.$$

If instead of $q = \rho + i\tau$ one chooses another value of q from the rectangle U , then:

1. $|q|$ is multiplied by a real number from the interval $[1-\delta, 1+\delta]$. Indeed, the numbers from U have positive real and imaginary parts. The maximal and minimal possible ratios of moduli of numbers from U are

$$|(\rho+i\tau+(1+i)\varepsilon)/(\rho+i\tau-(1+i)\varepsilon)|^{\pm 1}$$

from which the claim follows.

2. $|10q^{12}z^3|$ is multiplied by a number from the interval

$$[(1-\delta)^{12}, (1+\delta)^{12}] = [0.9999999628\dots, 1.000000036\dots];$$

this results directly from 1. (If we consider the monomial $15q^{18}z^4$, then such a change of q would multiply $|15q^{18}z^4|$ by a number from the interval $[(1-\delta)^{18}, (1+\delta)^{18}]$ etc.)

3. The argument of q can change by not more than

$$\arg((\rho-\varepsilon+i(\tau+\varepsilon))/(\rho+\varepsilon+i(\tau-\varepsilon))) = 1.091393649\dots \times 10^{-9}.$$

Clearly, such changes of q can change the real or the imaginary part of the sum of the first six monomials of θ^* (see (9)) by less than 10^{-7} .

4. If one has to change the value of z by any value from V , then one multiplies $|z|$ by a number from the interval $[|B/D|, |D/B|] = [0.9999922545\dots, 1.000007306\dots]$ while its argument changes by not more than $\arg(A/C) = 0.0006628745824\dots$. This can change the sum of the first six monomials of θ^* by a term whose modulus is less than 10^{-2} . The same is true if one changes simultaneously q and z .

Proof of Lemma 10. For $q \in U$ and $|z| = |q|^{-2}$ one has

$$|\theta_r - \theta_{(4)}| \leq \sum_{j=5}^n |q|^{j(j+1)/2} |q|^{-2j} = \sum_{j=5}^n |q|^{j(j-3)/2} \leq \sum_{j=5}^{\infty} |q|^{j(j-3)/2} < 0.02. \quad (18)$$

As for the polynomial $S := \theta_{(4)} = 1 + qz + q^3z^2 + q^6z^3 + q^{10}z^4$, when one sets $z := |q|^{-2}(\cos t + i \sin t)$, for $q = \rho + i\tau$ one gets

$$\begin{aligned} \operatorname{Re} S &= 1 + 2.127219493\dots \cos t - 0.601264628\dots \sin t \\ &\quad + 1.497715597\dots \cos(2t) - 1.625862822\dots \sin(2t) - 0.08191367962\dots \cos(3t) \\ &\quad - 0.9966394282\dots \sin(3t) - 0.1895137736\dots \cos(4t) - 0.07721972768\dots \sin(4t) \\ \operatorname{Im} S &= 2.127219493\dots \sin t + 0.6012646284\dots \cos t + 1.497715597\dots \sin(2t) \\ &\quad + 1.625862822\dots \cos(2t) - 0.08191367962\dots \sin(3t) + 0.9966394282\dots \cos(3t) \\ &\quad - 0.1895137736\dots \sin(4t) + 0.07721972768\dots \cos(4t) \end{aligned}$$

When one varies the values of q while remaining in the rectangle U one cannot change any of the coefficients of these trigonometric polynomials by more than 10^{-5} . Indeed,

1. The coefficient of $\cos(kt)$ or $\sin(kt)$ equals $\pm|q|^{-2k}\text{Re}(q^{k(k+1)/2})$ or $\pm|q|^{-2k}\text{Im}(q^{k(k+1)/2})$, $1 \leq k \leq 4$.
2. The possible variations of $q^{k(k+1)/2}$ (for $q \in U$) can be deduced from parts 1 and 3 of Example 13. The moduli of these numbers are < 0.46 .
3. The minimal and maximal possible values of $|q|^{-2k}$ for $q \in U$ are obtained for $q = \rho + i\tau \pm (1+i)\varepsilon$. For $k = 4$ they equal $570.1914944\dots$ and $570.1914999\dots$, i. e. the modulus of their difference is $< 6 \times 10^{-6}$. For $k = 1, 2$ and 3 this modulus is even smaller.

As the sum of the moduli of all coefficients of $\text{Re}S$ and $\text{Im}S$ is less than 50 and one has $0 \leq |\cos(kt)|, |\sin(kt)| \leq 1$, the values of $\text{Re}S$ and $\text{Im}S$ can vary by less than 10^{-3} when $q \in U$.

For $q = \rho + i\tau$ and for $t \in [0, 4]$ (resp. $t \in [4, 4.5]$ or $t \in [4.5, 5.5]$ or $t \in [5.5, 2\pi]$) one has $\text{Im}S > 0.04 > 0.02$ (resp. $\text{Re}S < -0.04$ or $\text{Im}S < -0.04$ or $\text{Re}S > 0.04$) hence $|S| > 0.04 > 0.02$. One can prove that the claimed inequalities hold true in the mentioned intervals by showing (say, using MAPLE) that the corresponding equalities have no solutions in these intervals and that for at least one point of the interval there is strict inequality. Hence for $q \in U$ and for $|z| = |q|^{-2}$ one has $|\theta_{(4)}| > 0.03 > 0.02 \geq |\theta_r - \theta_{(4)}|$. The lemma follows now from the Rouché theorem. \square

Proof of Lemma 12. Recall that equality (9) holds true. Our aim is to estimate the real and imaginary parts of the monomials in its right-hand side for $(q, z) \in U \times V$. First of all we list the values of the first several monomials (excluding the constant term 1) for $z = A$ or $z = C$ and for $q = \rho + i\tau$. The monomials and their values are:

$z = A$		$z = C$	
$3q^3z$	$6q^7z^2$	$10q^{12}z^3$	$15q^{18}z^4$
$21q^{25}z^5$			
$-2.368899634\dots$	$-i 0.06868680921\dots$	$-2.368835235\dots$	$-i 0.07025653317\dots$
$+1.600516792\dots$	$+i 0.554377995\dots$	$+1.599755638\dots$	$+i 0.5564907703\dots$
$-0.2788813462\dots$	$-i 0.3612445530\dots$	$-0.2781559523\dots$	$-i 0.3617900215\dots$
$-0.009845358519\dots$	$+i 0.0490842341\dots$	$-0.009975161466\dots$	$+i 0.0490564368\dots$
$+0.002251080781\dots$	$-i 0.0005556207520\dots$	$+0.002252822699\dots$	$-i 0.0005481355646\dots$

Observe that the sum of the real (resp. the imaginary) parts of these five monomials belongs to the interval $(0.05, 0.06)$ (resp. $(0.17, 0.18)$), for $z = A$ and for $z = C$.

Next we use Example 13. As $DR[3q^3z + \dots + 21q^{25}z^5] \leq DR[3q^3z] + \dots + DR[21q^{25}z^5]$, one can write

$$DR_1 := DR[3q^3z + \dots + 21q^{25}z^5] \leq 10^{-2} \quad \text{and} \quad DI_1 := DI[3q^3z + \dots + 21q^{25}z^5] \leq 10^{-2}. \quad (19)$$

Starting with $21q^{25}z^5$ the moduli of the monomials decrease faster than a geometric progression with ratio $1/10$. Therefore the sum of all other monomials (i. e. $28q^{33}z^6 + 36q^{42}z^7 + \dots$) contributes to the real and the imaginary part of θ^* by less than 10^{-4} . Thus for $(q, z) \in U \times V$ one has $DR[\theta^*] < 0.02$ and $DI[\theta^*] < 0.02$, i. e. $\text{Re}(\theta^*) \in (0.03, 0.08)$ and $\text{Im}(\theta^*) \in (0.15, 0.20)$. This proves part (1) of the lemma.

Remark 14. The small possible variation of $\arg z$ when $z \in V$ implies that for $q = \rho + i\tau$ the extremal possible values of the real and imaginary parts of the indicated monomials are attained for $z = A$ and $z = C$, the values of $z \in V$ with largest and smallest possible arguments.

To prove part (2) using the same scheme of reasoning we use equality (8). The first several monomials (excluding 1) and their values for $z = A$ or $z = C$ and for $q = \rho + i\tau$ are equal to:

$z = A$		$z = C$	
$6q^2z$	$18q^5z^2$	$40q^9z^3$	$75q^{14}z^4$
$126q^{20}z^5$	$196q^{27}z^6$		
$-10.16093686 \dots$	$+i 2.556447275 \dots$	$-10.16255051 \dots$	$+i 2.549691538 \dots$
$+24.24583379 \dots$	$-i 5.358044687 \dots$	$+24.25254021 \dots$	$-i 5.325813590 \dots$
$-19.64443131 \dots$	$-i 1.712625563 \dots$	$-19.64053031 \dots$	$-i 1.751646827 \dots$
$+4.696498002 \dots$	$+i 3.697047043 \dots$	$+4.686533589 \dots$	$+i 3.709371862 \dots$
$-0.03562046677 \dots$	$-i 0.7334753216 \dots$	$-0.03318797953 \dots$	$-i 0.7335609426 \dots$
$-0.03337082701 \dots$	$+i 0.01773395043 \dots$	$-0.03343954113 \dots$	$+i 0.01760026851 \dots$

The sums of the real (resp. imaginary) parts of these monomials and 1 belong to the interval $(0.06, 0.08)$ (resp. $(-1.57, -1.55)$). Starting with $196q^{27}z^6$, the moduli of the monomials decrease faster than a geometric progression with ratio $1/10$.

We established in the proof of part (1) inequalities (19) using inequality (10). Compare the monomials $3q^3z$, ..., $21q^{25}z^5$ with the monomials $18q^5z^2$, ..., $196q^{27}z^6$. The respective numerical coefficients increase less than 10 times, the degree of q is larger by 2 and the one of z is larger by 1. The maximal possible value of $|q^2z|$ in $U \times V$ is < 1.8 . Therefore

$$\begin{aligned} DR_2 &:= DR[18q^5z^2 + \dots + 196q^{27}z^6] \leq 10 \times 1.8 \times (DR_1 + DI_1) \leq 0.72 \quad \text{and} \\ DI_2 &:= DI[18q^5z^2 + \dots + 196q^{27}z^6] \leq 10 \times 1.8 \times (DR_1 + DI_1) \leq 0.72 . \end{aligned}$$

Consider $\theta^0 := \theta_{qz} - (\theta_{qz})_{(6)} = 288q^{35}z^7 + 405q^{44}z^8 + \dots$. It is true that $DR[6q^2z] < 10^{-2}$, $DI[6q^2z] < 10^{-2}$, $DR[\theta^0] \leq 10^{-3}$ and $DI[\theta^0] \leq 10^{-3}$. Hence $DR[\theta_{qz}] < 0.76$ and $DI[\theta_{qz}] < 0.76$ from which part (2) of the lemma follows. \square

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